Q1. (30 points; 10, 10, 10 points) Solve the following recurrence relations, expressing them using Θ() notation.

a. $f(n) = 10f(n-1) - 25f(n-2)$ for $n \geq 2$; $f(0) = 3, f(1) = 17.$

$x^2 - 10x + 25 = 0$

$\boldsymbol{r1 = r2 = 5}$

$\boldsymbol{f(n) = 5^n c1 + 5^n n c2}$

$\boldsymbol{f(0) = 5^n c1 + 2^n n c2 = 3 = c_1}$

$\boldsymbol{f(1) = 5^n c1 + 5^n n c2 = 17 = 5c_1 + 5c_2}$

$\boldsymbol{c1 = 3,\ c2 = 2/5}$

$\boldsymbol{f(n) = 3\ 5^n + 5^n n = \theta(n\ 5^n)}$

b. Solve the following recurrence relations using the Master theorem?

i. $f(n) = 4f\left(\frac{n}{16}\right) + \sqrt{n}$ *for* $n \geq 2$; $f(0) = 1, f(1) = 0.$

Using master theorem:

a=4 , b=16 and $\boldsymbol{n^{(\log_{16} 4)} = n^{(1/2)}}$

$\boldsymbol{g(n) = n^{(1/2)} = \Theta(n^{(1/2)})}$ **.**

It follows that:

$\boldsymbol{f(n) = \Theta(n^{(1/2)} \log n)}$**.**

c. $f(n) = 2 f(n/2) + n^2$ for $n \geq 2$; $f(0) = 6; f(1) = 1$;

Using master theorem:

a=2 , b=2 and $n^{(\log_2 2)}=n$

$g(n) = n^2 = \Omega(n^{1+1})$.

Regularity Condition:

$2\, g(n/2) = 2\,(n/2)^2 = 0.5\, n^2 \leq c\, g(n) = c.\, n^2$, Obviously, choosing $c = 0.5 < 1$ the inequality holds for all $n \geq 1 = n_0$.

It follows that the conditions for the third case of the Master theorem is satisfied, and hence:

$f(n) = \Theta(n^2)$.

Q2. (15 points) Use Horner's rule to evaluate the following polynomial at $x = -6$.

$$p(x) = 5x^9 + 29x^8 + 4x^7 + 65x^6 + 31x^5 - 40x^3 - 24x^2 + 8x + 40$$

Note: A final answer without using the algorithm is worth zero points.

p = 5
p = 5(-6) + 29 = -1
p = -1(-6) + 4 = 10
p = 10(-6) + 65 = 5
p = 5(-6) + 31 = 1
p = 1(-6) + 0 = -6
p = -6(-6) - 40 = -4
p = -4(-6) - 24 = 0
p = 0(-6) + 8 = 8
p = 8(-6) + 40 = -8

Q2. (20 points)

a. (10 points) Find the solution to the following recurrence relation in terms of Big $\Theta()$ notation.

$$f(n) = 4f\left(\frac{n}{2}\right) + n^2 \log n \qquad f(1) = 2$$

Using the master theorem is not an option as the relationship between n^2 and $n^2 \log n$ does not fall in any of the three cases. Hence, we resort to expansion:

$$f(n) = 4f\left(\frac{n}{2}\right) + n^2 \log n$$
$$= 4\left(4f\left(\frac{n}{2^2}\right) + \frac{n^2}{2^2}\log\frac{n}{2}\right) + n^2 \log n$$
$$= 4^2 f\left(\frac{n}{2^2}\right) + n^2 \log\frac{n}{2} + n^2 \log n$$
$$= 4^2\left(4f\left(\frac{n}{2^3}\right) + \frac{n^2}{2^4}\log\frac{n}{2^2}\right) + n^2 \log\frac{n}{2} + n^2 \log n$$
$$= 4^3 f\left(\frac{n}{2^3}\right) + n^2 \log\frac{n}{2^2} + n^2 \log\frac{n}{2} + n^2 \log n$$

+3

$$= \cdots$$
$$= 4^k f\left(\frac{n}{2^k}\right) + n^2 \log\frac{n}{2^{k-1}} + n^2 \log\frac{n}{2^{k-2}} + \cdots + n^2 \log n$$

+2

$$= \cdots$$
$$= 4^{\log n} f(1) + n^2 \log\frac{n}{2^{\log n - 1}} + n^2 \log\frac{n}{2^{\log n - 2}} + \cdots + n^2 \log\frac{n}{2} + n^2 \log n$$

+1

$$= 2n^2 + n^2\left(\log 2 + \log 2^2 + \cdots + \log 2^{\log n}\right)$$
$$= 2n^2 + n^2 \sum_{i=1}^{\log n} \log 2^i$$
$$= 2n^2 + n^2 \sum_{i=1}^{\log n} i$$

+2

$$= 2n^2 + \frac{n^2\left(\log n\,(\log n + 1)\right)}{2}$$
$$= \Theta(n^2 \log^2 n)$$

+1

b. (10 points) Use the change of variable method to find the solution to the following recurrence relation in terms of Big $\Theta()$ notation.

$$T(n) = 4T\left(n^{\frac{2}{3}}\right) + \log^2 n \qquad T(1) = T(2) = 3$$

Let $n = 2^k$, (i.e., $k = \log n$). Then, +2

$$T(2^k) = 4T(2^k)^{\frac{2}{3}} + \log^2 2^k$$

$$= 4T\left(2^{\frac{2k}{3}}\right) + k^2$$ +1

Now, consider the change of domain $T(2^k) \leftrightarrow f(k)$:

$$f(k) = 4f\left(\frac{2k}{3}\right) + k^2$$ +2

$$= 4f\left(\frac{k}{3/2}\right) + k^2$$

With $a = 4, b = \frac{3}{2}, k^{\log_{\frac{3}{2}} 4}$ and $g(k) = k^2$, +1

we see that $g(k) = k^2 = O\left(k^{\log_{\frac{3}{2}} 4 - \epsilon}\right)$, with $\epsilon = 1$. +2

Hence, the first case of the master theorem holds, and therefore,

$$f(k) = \Theta\left(k^{\log_{\frac{3}{2}} 4}\right)$$ +2

Changing back to n, we find that $T(n) = \Theta\left(\log^{\log_{\frac{3}{2}} 4} n\right)$